Question 16 **(10 marks)**

An automated milk bottling machine fills bottles uniformly to between 247 ml and 255 ml. The label on the bottle states that it holds 250 ml.

(a) Determine the probability that a bottle selected randomly from the conveyor belt of this machine contains less than the labelled amount. (3 marks)

(b) Calculate the mean and standard deviation of the amount of milk in the bottles. (4 marks)

A worker selects bottles from the conveyor belt, one at a time.

(c) Determine the probability that it takes the selection of 15 bottles before five bottles containing less than the labelled amount have been selected. (3 marks)

Question 17 **(7 marks)**

A school has analysed the examination scores for all its Year 12 students taking Methods as a subject. Let X = the examination percentage scores of all the Methods Year 12 students at the school. The school found that the mean was 75 with a standard deviation of 22.

Determine the following.

(a) $E(X + 5)$ (1 mark)

(b) $Var(25 - 2X)$ (2 marks)

The school has decided to scale the results using the transformation $Y = aX + b$ where a and b are constants and Y = the scaled percentage scores. The aim is to change the mean to 60 and the standard deviation to 15.

(c) Determine the values of a and b. (4 marks)

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Section One: Calculator-free **35% (52 Marks)**

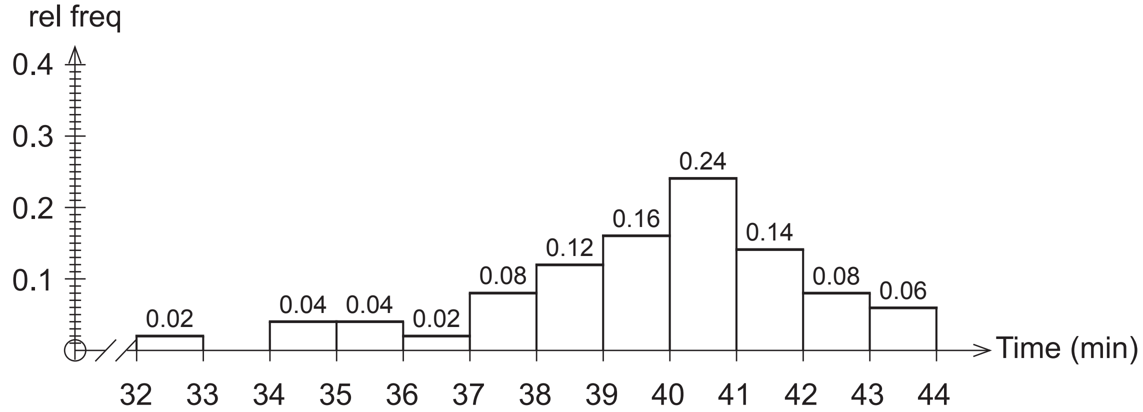
This section has **nine (9)** questions. Answer **all** questions. Write your answers in the spaces provided.

Supplementary pages for the use of planning/continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

Working time: 50 minutes.

Question 1 **(5 marks)**

Anastasia is a university student. She records the time it takes for her to get from home to her campus each day. The histogram of relative frequencies below shows the journey times she recorded.



Use the above data to estimate the probability of her next journey from home to her university campus

(a) taking her less than 36 minutes. (1 mark)

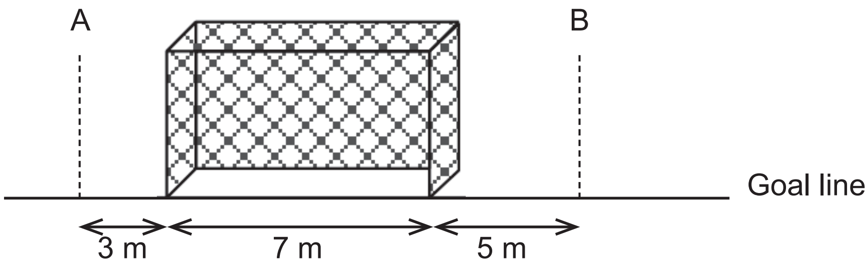
(b) taking at least 35 minutes but no more than 39 minutes. (2 marks)

On three consecutive days, Anastasia needs to be on campus no later than 10 am.

(c) If she leaves her home at 9:22 am each day, use the above data to estimate the probability that she makes it on or before time on all three days. (2 marks)

Question 2 **(6 marks)**

Michelle is a soccer goalkeeper and has built a machine to help her practise. The machine will shoot a soccer ball randomly along the ground at or near a goal that is seven metres wide. The machine is equally likely to shoot the ball so that the centre of the ball crosses the goal line anywhere between point A three metres left of the goal, and point B five metres right of the goal, as shown in the diagram below.



Michelle sets up a trial run without anyone in the goals. Assume the goal posts are of negligible width.

Let the random variable X be the distance the centre of the ball crosses the goal line to the right of point A.

(a) Complete the graphical representation of the probability density function for the random variable X. (2 marks)

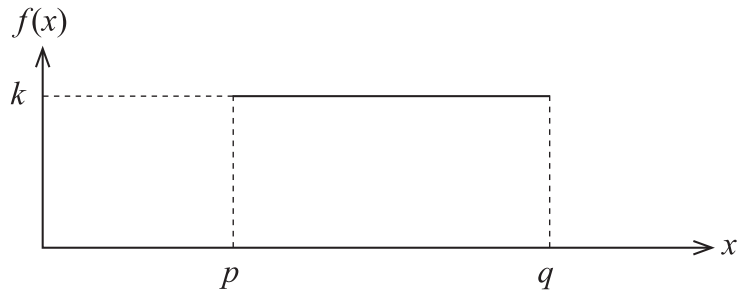
(b) What is the probability that the machine shoots a ball so that its centre misses the goal to the left? (1 mark)

(c) What is the probability that the machine shoots a ball so that its centre is inside the goal? (1 mark)

(d) If the machine shoots a ball so that its centre misses the goal, what is the probability that the ball's centre misses to the right? (2 marks)

Question 2 **(10 marks)**

It takes Nahyun between 15 and 40 minutes to get to school each day, depending on traffic conditions. Nahyun leaves home for school at 8.00 am each school day. Let the random variable X be the time, in minutes after 8:00 am, that Nahyun arrives at school. The probability density function of X is shown below.



(a) What is the name of this type of distribution? (1 mark)

(b) Determine:

(i) the values of p, q and k (2 marks)

(ii) the expected value of X (1 mark)

(iii) the probability that Nahyun arrives at school before 8:25 am. (2 marks)

Nahyun will be late for her first class if she arrives at school after 8:28 am. Otherwise, she will not be late.

(c) If Nahyun is not late for her first class, what is the probability that she arrives after 8:25 am? (2 marks)

(d) If Nahyun only wants to be late for her first class at most 4% of the time, what time should she leave home, assuming the 15 to 40 minute travel time remains the same? (2 marks)

Question 3 **(3 marks)**

Given that $\ln(2) \approx 0.693$, use the increments formula to determine an approximation for $\ln(2.02)$.

Question 11 **(9 marks)**

A pizza shop estimates that the time X hours to deliver a pizza from when it is ordered is a continuous random variable with probability density function given by

$$f(x) = \begin{cases} \frac{4}{3} - \frac{2}{3}x & , \ 0 < x < 1 \\ 0 & , \ \text{otherwise.} \end{cases}$$

(a) What is the probability of a pizza being delivered within half an hour of being ordered? (2 marks)

(b) Calculate the mean delivery time to the nearest minute. (3 marks)

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(c) Calculate the standard deviation of the delivery time to the nearest minute. (4 marks)

Question 10 **(7 marks)**

The following function is a probability density function on the given interval:

$$f(x) = \begin{cases} ax^2(x-2) & \text{for } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

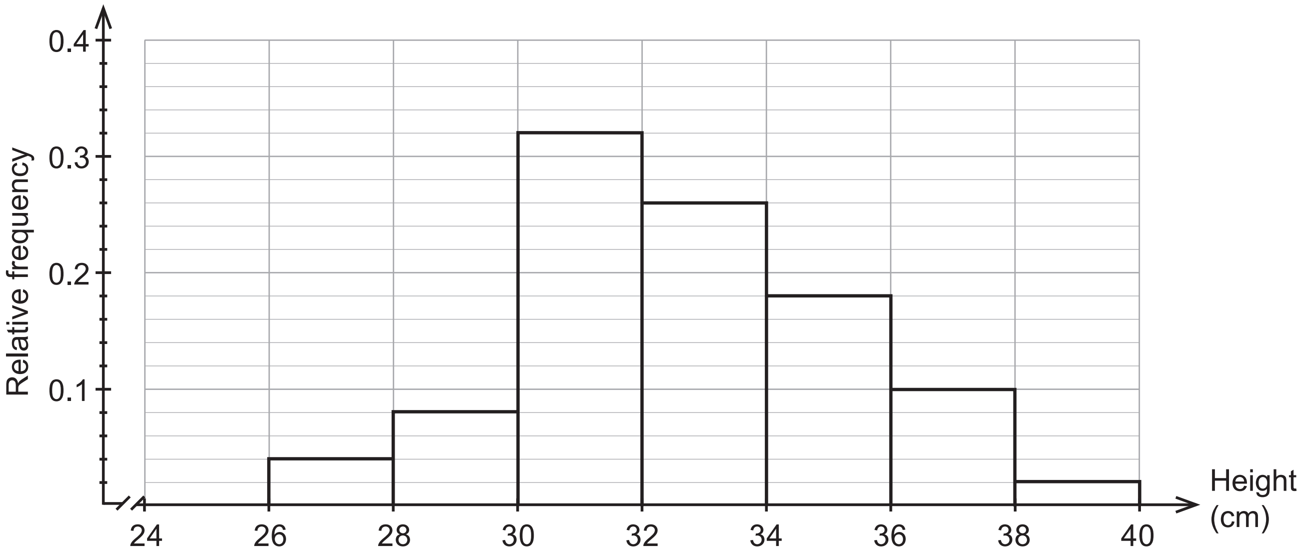
(a) Find the value of a. (3 marks)

(b) Find the probability that $x \ge 1.2$ (2 marks)

(c) Find the median of the distribution. (2 marks)

Question 4 **(9 marks)**

The heights reached by a species of small plant at maturity are measured by a team of biologists. The results are shown in the histogram of relative frequencies below.



(a) Determine the probability that a mature plant of this species reaches no higher than 30 cm. (1 mark)

(b) If a mature plant reaches a height of at least 32 cm, what is the probability that its height reaches above 38 cm? (2 marks)

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Another team of biologists is studying the mature heights of a species of hedge. The height, h metres, has a probability density function, $d(h)$, as given below.

$$d(h) = \begin{cases} \dfrac{h-1}{5} & \text{for} \quad 1 \le h \le 2 \\ kh^2 & \text{for} \quad 2 < h \le 4 \\ 0 & \quad\quad \text{otherwise} \end{cases}$$

(c) What percentage of hedges from this study reaches a mature height less than 2 m? (3 marks)

(d) Determine the value of k. (3 marks)

Question 6 **(10 marks)**

The error X in digitising a communication signal has a uniform distribution with probability density function given by

$$f(x) = \begin{cases} 1, & -0.5 < x < 0.5, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Sketch the graph of $f(x)$. (2 marks)

(b) What is the probability that the error is at least 0.35? (1 mark)

(c) If the error is negative, what is the probability that it is less than –0.35? (2 marks)

(d) An engineer is more interested in the square of the error. What is the probability that the square of the error is less than 0.09? (2 marks)

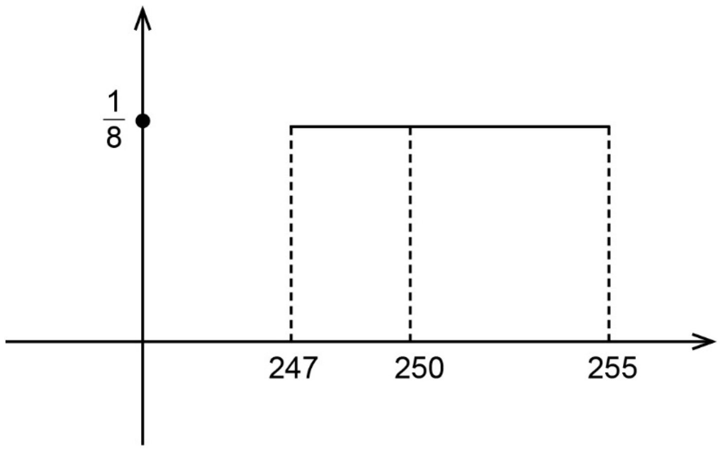
(e) Calculate the variance of the error. (3 marks)

Question 16 **(10 marks)**

An automated milk bottling machine fills bottles uniformly to between 247 ml and 255 ml. The label on the bottle states that it holds 250 ml.

(a) Determine the probability that a bottle selected randomly from the conveyor belt of this machine contains less than the labelled amount. (3 marks)

Solution



$P(X < 250) = \dfrac{3}{8}$

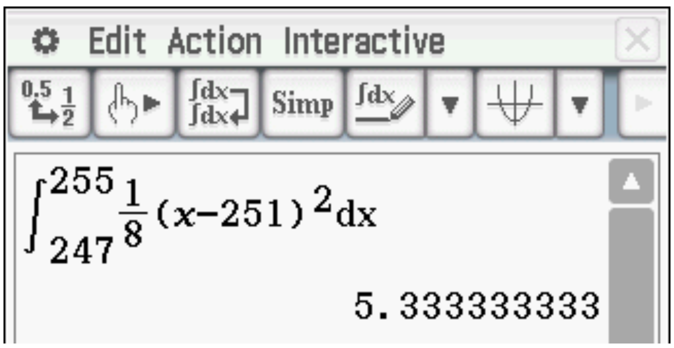
Specific behaviours

✓ evaluates pdf value
✓ uses area to calculate probability
✓ determines probability

(b) Calculate the mean and standard deviation of the amount of milk in the bottles. (4 marks)

Solution

Mean = $\dfrac{255+247}{2} = 251$



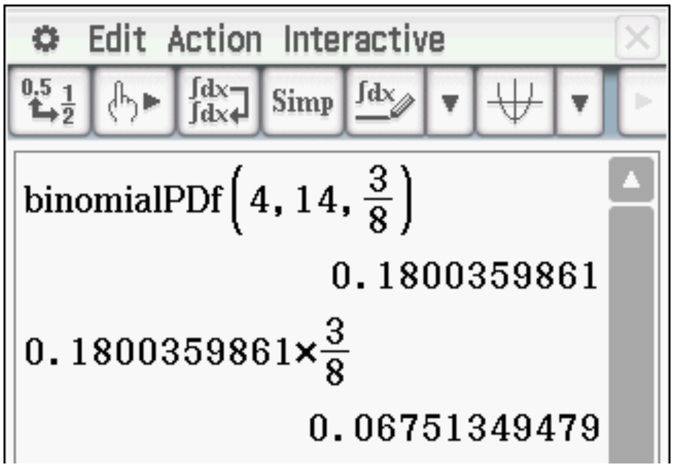
Standard deviation = $\sqrt{Var(x)}$ = 2.3094

Specific behaviours

✓ determines mean
✓ uses appropriate pdf function
✓ sets up an integral to find variance
✓ determines standard deviation

A worker selects bottles from the conveyor belt, one at a time.

(c) Determine the probability that it takes the selection of 15 bottles before five bottles containing less than the labelled amount have been selected. (3 marks)

Solution


Specific behaviours
✓ uses binomial distribution for first 14 selections ✓ determines probability for the first four bottles ✓ determines final probability

Question 17 **(7 marks)**

A school has analysed the examination scores for all its Year 12 students taking Methods as a subject. Let X = the examination percentage scores of all the Methods Year 12 students at the school. The school found that the mean was 75 with a standard deviation of 22.

Determine the following.

(a) $E(X+5)$ (1 mark)

Solution
$E(X+5)=E(X)+5=80$
Specific behaviours
✓ determines mean

(b) $Var(25-2X)$ (2 marks)

Solution
$Var(25-2X)=2^2 Var(X)=4\times 22\times 22=1936$
Specific behaviours
✓ uses a positive factor of four ✓ determines variance

The school has decided to scale the results using the transformation $Y=aX+b$ where a and b are constants and Y = the scaled percentage scores. The aim is to change the mean to 60 and the standard deviation to 15.

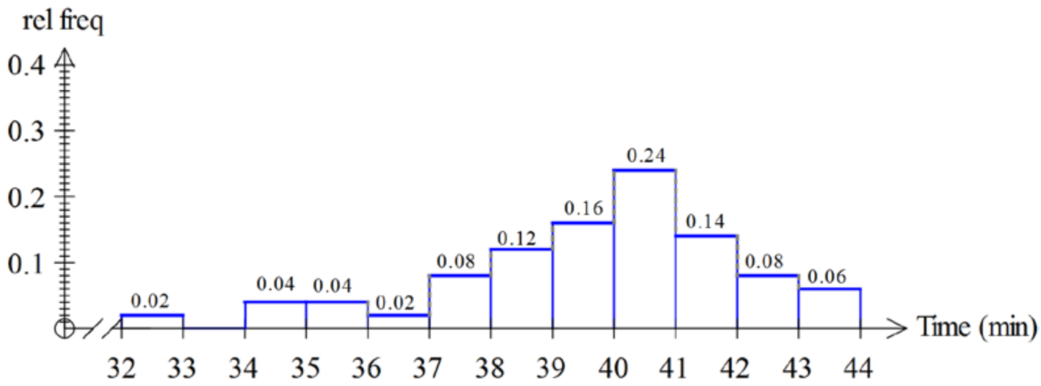
(c) Determine the values of a and b. (4 marks)

Solution
$15=22a$ $a=\frac{15}{22}\approx 0.682$ $60=75a+b$ $b=\frac{195}{22}\approx 8.864$
Specific behaviours
✓ determines change on standard deviation first ✓ sets up at least one equations for a and b ✓ determines a ✓ determines b

Section One: Calculator-free **35% (52 Marks)**

Question 1 **(5 marks)**

Anastasia is a university student. She records the time it takes for her to get from home to her campus each day. The histogram of relative frequencies below shows the journey times she recorded.



Use the above data to estimate the probability of her next journey from home to her university campus

(a) taking her less than 36 minutes. (1 mark)

Solution
$P(T \leq 36) = 0.02 + 0.04 + 0.04$ $= 0.1$
Specific behaviours
✓ sums relative frequencies to determine probability

(b) taking at least 35 minutes but no more than 39 minutes. (2 marks)

Solution
$P(35 \leq T \leq 39) = 0.04 + 0.02 + 0.08 + 0.12$ $= 0.26$
Specific behaviours
✓ recognises the probability involves frequencies above 35 and below 39 ✓ sums relative frequencies to determine probability

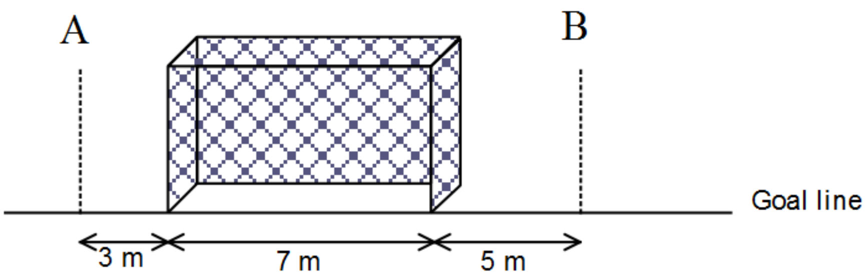
On three consecutive days, Anastasia needs to be on campus no later than 10 am.

(c) If she leaves her home at 9.22 am each day, use the above data to estimate the probability that she makes it on or before time on all three days. (2 marks)

Solution
$P(T \leq 38) = 0.02 + 0.04 + 0.04 + 0.02 + 0.08$ $= 0.2$ 3 consecutive days $= 0.2^3$ $= 0.008$
Specific behaviours
✓ sums relative frequencies to determine probability ✓ determines probability of 3 consecutive days

Question 2 **(6 marks)**

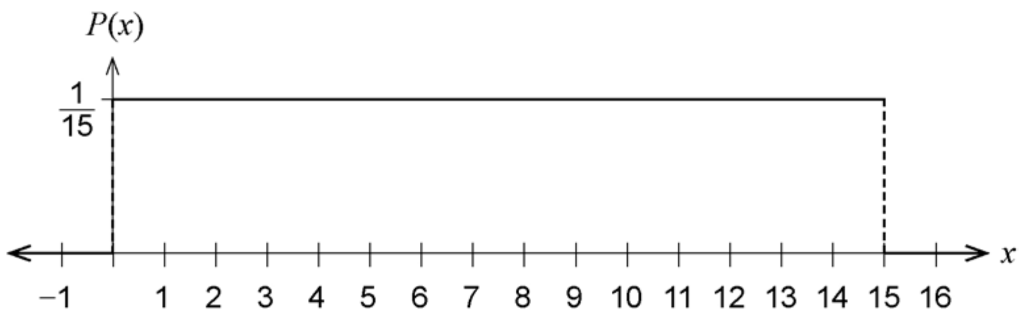
Michelle is a soccer goalkeeper and has built a machine to help her practise. The machine will shoot a soccer ball randomly along the ground at or near a goal that is seven metres wide. The machine is equally likely to shoot the ball so that the centre of the ball crosses the goal line anywhere between point A three metres left of the goal, and point B five metres right of the goal, as shown in the diagram below.



Michelle sets up a trial run without anyone in the goals. Assume the goal posts are of negligible width.

Let the random variable X be the distance the centre of the ball crosses the goal line to the right of point A.

(a) Complete the graphical representation of the probability density function for the random variable X. (2 marks)

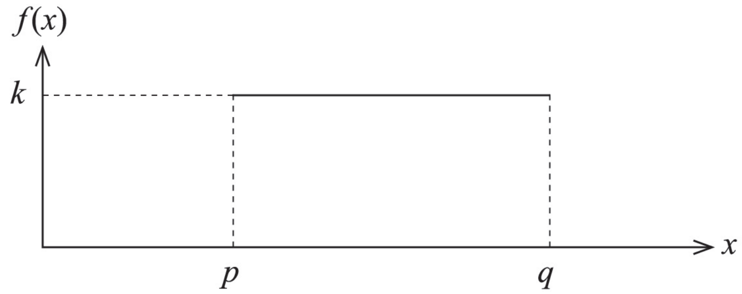
Solution

Specific behaviours
✓ draws uniform distribution correctly ✓ includes $P(x)$ axes scale

(b) What is the probability that the machine shoots a ball so that its centre misses the goal to the left? (1 mark)

Solution
$\dfrac{3}{15}$
Specific behaviours
✓ states correct probability

Question 2 **(10 marks)**

It takes Nahyun between 15 and 40 minutes to get to school each day, depending on traffic conditions. Nahyun leaves home for school at 8.00 am each school day. Let the random variable X be the time, in minutes after 8:00 am, that Nahyun arrives at school. The probability density function of X is shown below.



(a) What is the name of this type of distribution? (1 mark)

Solution
Continuous uniform distribution
Specific behaviours
✓ correctly states the name of the distribution

(b) Determine:

(i) the values of p, q and k (2 marks)

Solution
$p = 15$ $q = 40$ $k = \frac{1}{25}$
Specific behaviours
✓ correctly states the values of p and q ✓ correctly states the value of k

(ii) the expected value of X (1 mark)

Solution
$E(x) = \frac{40+15}{2}$ $= 27.5 \text{ minutes}$
Specific behaviours
✓ correctly states the expected value

Question 2 (continued)

(iii) the probability that Nahyun arrives at school before 8:25 am. (2 marks)

Solution
$\mathrm{P}(X < 25) = \dfrac{25-15}{25}$ $= \dfrac{10}{25}\left\{= \dfrac{2}{5}\right\}$
Specific behaviours
✓ identifies the area between 15 and 25 is required   ✓ calculates the correct probability (simplified probability not required)

Nahyun will be late for her first class if she arrives at school after 8:28 am. Otherwise, she will not be late.

(c) If Nahyun is not late for her first class, what is the probability that she arrives after 8:25 am? (2 marks)

Solution
$P(X > 25 \mid X < 28) = \dfrac{3}{13}$
Specific behaviours
✓ correctly identifies the situation is a conditional probability   ✓ determines the correct probability

(d) If Nahyun only wants to be late for her first class at most 4% of the time, what time should she leave home, assuming the 15 to 40 minute travel time remains the same? (2 marks)

Solution
$4\% = \dfrac{4}{100} = \dfrac{1}{25}$ $\therefore$ leaves 39 minutes before 8:28 am   She should leave home at 7:49 am
Specific behaviours
✓ determines 4% = a probability of $\dfrac{1}{25}$ ✓ correctly determines the time

Question 2 (continued)

(c) What is the probability that the machine shoots a ball so that its centre is inside the goal? (1 mark)

Solution
$\frac{7}{15}$
Specific behaviours
✓ states correct probability

(d) If the machine shoots a ball so that its centre misses the goal, what is the probability that the ball's centre misses to the right? (2 marks)

Solution
$\frac{\frac{5}{15}}{\frac{8}{15}} = \frac{5}{8}$
Specific behaviours
✓ correctly determines numerator ✓ correctly determines denominator

Question 11 **(9 marks)**

A pizza shop estimates that the time X hours to deliver a pizza from when it is ordered is a continuous random variable with probability density function given by

$$f(x) = \begin{cases} \frac{4}{3} - \frac{2}{3}x \ , & 0 < x < 1 \\ 0 \ , & \text{otherwise.} \end{cases}$$

(a) What is the probability of a pizza being delivered within half an hour of being ordered? (2 marks)

Solution
$P(X < 0.5) = \int_0^{0.5} \left(\frac{4}{3} - \frac{2}{3}x\right) dx = \frac{4}{3}x - \frac{1}{3}x^2 \Big\|_0^{0.5} = \frac{2}{3} - \frac{1}{12} = \frac{7}{12} \approx 0.5833$ OR $P(X < 0.5) = \text{Area of trapezium} = \frac{1}{4}\left(\frac{4}{3} + 1\right) = \frac{7}{12} \approx 0.5833$
Specific behaviours
✓ writes correct integral (or area) expression for probability   ✓ calculates probability correctly

(b) Calculate the mean delivery time to the nearest minute. (3 marks)

Solution
$E(X) = \int_0^1 x\left(\frac{4}{3} - \frac{2}{3}x\right) dx = \frac{2}{3}x^2 - \frac{2}{9}x^3 \Big\|_0^1 = \frac{2}{3} - \frac{2}{9} = \frac{4}{9} \approx 0.4444$ That is, 27 minutes.
Specific behaviours
✓ writes the correct integral for the mean   ✓ calculates the mean correctly   ✓ converts to minutes

Question 11 (continued)

(c) Calculate the standard deviation of the delivery time to the nearest minute. (4 marks)

Solution
$\text{Var}(X) = \int_0^1 \left(\frac{4}{3} - \frac{2}{3}x\right)\left(x - \frac{4}{9}\right)^2 dx$ $= 0.0802$ OR $E\left(X^2\right) = \int_0^1 x^2\left(\frac{4}{3} - \frac{2}{3}x\right)dx = \frac{4}{9}x^3 - \frac{1}{6}x^4\Big\vert_0^1 = \frac{4}{9} - \frac{1}{6} = \frac{10}{36} = \frac{5}{18} \approx 0.2778$ So $\text{Var}\left(X\right) = \frac{5}{18} - \frac{16}{81} = \frac{13}{162} \approx 0.0802$ $\sigma = \sqrt{0.0802} \approx 0.2833$ That is, 17 minutes.
Specific behaviours
✓ calculates $E(X^2)$ correctly or states integral for VAR   ✓ calculates the variance correctly   ✓ calculates standard deviation correctly   ✓ converts to minutes

Question 10 **(7 marks)**

The following function is a probability density function on the given interval:

$$f(x)=\begin{cases} ax^2(x-2) & \text{for } 0 \le x \le 2. \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of a. (3 marks)

Solution
If pdf on domain then $\int_0^2 f(x)dx = 1$ $\int_0^2 f(x)\,dx = 1$ $\int_0^2 ax^2(x-2)\,dx = -\frac{4a}{3}$ $\therefore -\frac{4a}{3} = 1$ $\therefore a = -\frac{3}{4}$
Specific behaviours
✓ uses integration for domain =1 ✓ calculates integration ✓ finds a

(b) Find the probability that $x \ge 1\cdot 2$. (2 marks)

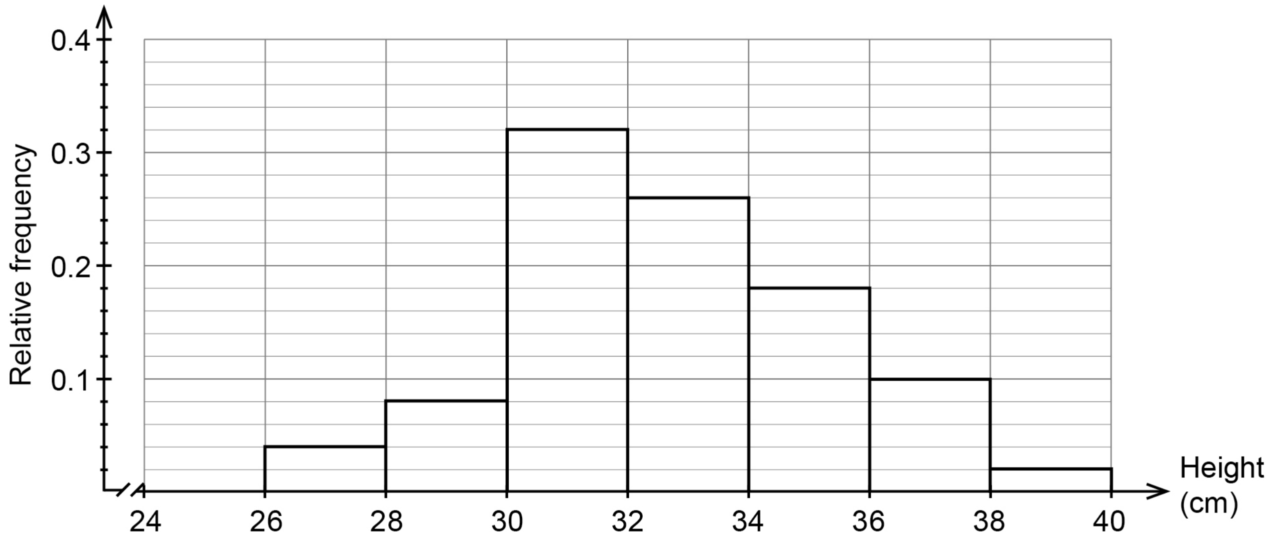
Solution
$\int_{1.2}^{2} \frac{-3x^2(x-2)}{4}\,dx$ $=0.5248$
Specific behaviours
✓ uses correct integral ✓ calculates probability

(c) Find the median of the distribution. (2 marks)

Solution
Solve $\int_0^m f(x)\,dx = 0.5$ over domain $0 \le x \le 2$ $\int_0^m f(x)dx = -\frac{3m^4}{16} + \frac{m^3}{2}$ for median: $-\frac{3m^4}{16} + \frac{m^3}{2} = 0.5$ $m = 1.2285$
Specific behaviours
✓ uses correct integral ✓ determines $m = 1.2285$

Question 4 **(9 marks)**

The heights reached by a species of small plant at maturity are measured by a team of biologists. The results are shown in the histogram of relative frequencies below.



(a) Determine the probability that a mature plant of this species reaches no higher than 30 cm. (1 mark)

Solution
$P(h \le 30) = 0.04 + 0.08$ $= 0.12$
Specific behaviours
✓ determines the correct probability

(b) If a mature plant reaches a height of at least 32 cm, what is the probability that its height reaches above 38 cm? (2 marks)

Solution
$P(h \ge 38 \mid h \ge 32) = \dfrac{0.02}{0.56} = \dfrac{1}{28}$
Specific behaviours
✓ recognises conditional probability and determines the correct denominator of the conditional probability   ✓ determines the correct probability as a fraction

Question 4 (continued)

Another team of biologists is studying the mature heights of a species of hedge. The height, h metres, has a probability density function, $d(h)$, as given below.

$$d(h)=\begin{cases}\dfrac{h-1}{5} & \text{for} \quad 1\le h\le 2\\ kh^2 & \text{for} \quad 2<h\le 4\\ 0 & \text{otherwise}\end{cases}$$

(c) What percentage of hedges from this study reaches a mature height less than 2 m? (3 marks)

Solution
Probability of $1<h\le 2$: $\int_1^2 \frac{h-1}{5}\,dh=\frac{1}{5}\left[\frac{h^2}{2}-h\right]_1^2$ $=\frac{1}{5}\left[2-2-\frac{1}{2}+1\right]$ $=\frac{1}{10}$ 10% reach a height of less than 2 m
Specific behaviours
✓ recognises the need to integrate the first equation of the PDF from 1 to 2 ✓ antidifferentiates the first equation correctly ✓ determines the correct percentage

(d) Determine the value of k. (3 marks)

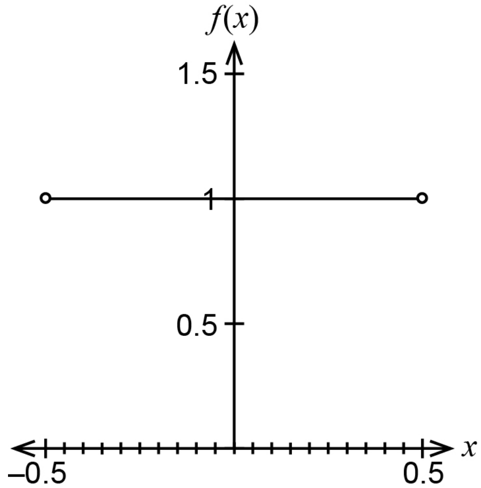
Solution
Probability of $2<h\le 4=1-\frac{1}{10}=\frac{9}{10}$ $\therefore \frac{9}{10}=\int_2^4 kh^2\,dh$ $\frac{9}{10}=k\left[\frac{h^3}{3}\right]_2^4$ $\frac{9}{10}=k\left[\frac{64-8}{3}\right]$ $k=\frac{27}{560}$
Specific behaviours
✓ recognises the need to integrate the second equation of the PDF from 2 to 4 and equates to the complement of part (c), $\frac{9}{10}$ ✓ antidifferentiates the second equation correctly ✓ determines the value of k

Question 6 **(10 marks)**

The error X in digitising a communication signal has a uniform distribution with probability density function given by

$$f(x) = \begin{cases} 1, & -0.5 < x < 0.5, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Sketch the graph of $f(x)$. (2 marks)

Solution

Specific behaviours
✓ draws horizontal line at y=1 ✓ indicates a domain of $-0.5 < x < 0.5$

(b) What is the probability that the error is at least 0.35? (1 mark)

Solution
$P(X > 0.35) = Area = 0.15$
Specific behaviours
✓ computes the correct probability

(c) If the error is negative, what is the probability that it is less than –0.35? (2 marks)

Solution
$P(X < -0.35 \vert X < 0) = \dfrac{P(X < -0.35 \cap X < 0)}{P(X < 0)} = \dfrac{P(X < -0.35)}{P(X < 0)} = \dfrac{0.15}{0.5} = 0.3$
Specific behaviours
✓ writes the correct conditional probability statement ✓ computes the probability correctly

(d) An engineer is more interested in the square of the error. What is the probability that the square of the error is less than 0.09? (2 marks)

Solution
$P(X^2 < 0.09) = P(-0.3 < X < 0.3) = 0.6$
Specific behaviours
✓ correctly expresses the required probability in terms of X ✓ computes the probability correctly

Question 6 (continued)

(e) Calculate the variance of the error. (3 marks)

Solution
$$E(X) = \int_{-0.5}^{0.5} x\,dx = 0$$ So $$Var(X) = \int_{-0.5}^{0.5} (x-0)^2(1)dx = \left.\frac{x^3}{3}\right\|_{-0.5}^{0.5} = \frac{0.125+0.125}{3} = \frac{1}{12}$$
Specific behaviours
✓ computes mean correctly ✓ states an integral for the variance ✓ evaluates the integral to determine variance correctly